

Alternative approach to the regularization of odd dimensional AdS gravity

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Abstract

In this paper I present an action principle for odd dimensional AdS gravity which consists of introducing another manifold with the same boundary and a very specific boundary term. This new action allows and alternative approach to the regularization of the theory, yielding a finite euclidean action and finite conserved charges.

The choice of the boundary term is justified on the grounds that an enhanced 'almost off-shell' local AdS/Conformal symmetry arises for that very special choice. One may say that the boundary term is dictated by a guiding symmetry principle.

Two sets of boundary conditions are considered, which yield regularization procedures analogous to (but different from) the standard 'background subtraction' and 'counterterms' regularization methods.

The Noether charges are constructed in general. As an application it is shown that for Schwarzschild-AdS black holes the charge associated to the time-like Killing vector is finite and is indeed the mass.

The Euclidean action for Schwarzschild-AdS black holes is computed, and it turns out to be finite, and to yield the right thermodynamics.

The previous paragraph may be interpreted in the sense that the boundary term dictated by the symmetry principle is the one that correctly regularizes the action.

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1 Introduction

The AdS/CFT correspondence [1, 2, 3]¹ has originated a great interest in AdS gravities for asymptotically AdS space-times. Semi-classical saddle point calculations in the gravity side would correspond to strong coupling properties of the dual conformal theory.

As the gravitational action diverges, suitable regularization methods are required in order to obtain sensible results.

There are essentially two main approaches to the regularization for the case of the standard Dirichlet boundary conditions, the 'background subtraction'[5, 6, 3] method and the 'counterterms' method [7, 8, 9, 10, 11, 12].

In this paper we present an alternative formalism for the regularization of General Relativity with a cosmological constant (also called 'AdS gravity') in odd dimensional space-times. To that end I consider an action functional for odd dimensional AdS gravity which consists of introducing another manifold with the same boundary and a very specific boundary term. Two possible sets of boundary conditions will be considered, different from the standard Dirichlet boundary conditions, which lead to action principles for which the action is regularized in a way analogous to the 'counterterms' and to the 'background subtraction' methods respectively, but differing from those methods in several aspects.

On the 'counterterms' side it is shown that the action and formalism of ref.[13] can be recovered as a particular case. The approach of ref.[13] has been studied and developed further in refs.[14, 15], where among other things the relationship with the above mentioned standard counterterms approach is discussed. To avoid confusion between the approach of refs.[13, 14, 15] and the counterterms of refs.[7, 8, 9, 10, 11, 12] R. Olea used the word 'kounterterms' to describe the former, a convention that I will follow here.

The construction proposed in this paper also makes possible an approach to the regularization in the spirit of the background subtraction methods, which however is not the same as the Hawking-Page approach, as it will be shown below.

It is worthwhile to mention that the problem of regularizing AdS gravity by introducing an action principle with suitable boundary terms and non Dirichlet boundary conditions has been solved in the even dimensional case in refs.[16, 17].

The idea of the alternative 'kounterterms' approach to the regularization of odd-dimensional AdS gravity introduced in ref.[13, 14, 15] was to 'borrow' the boundary term used to regularize Chern-Simons gravities² in ref.[19], with a very specific relative coefficient between the standard AdS gravity bulk term and the 'borrowed' boundary term, and using the same boundary conditions. Perhaps surprisingly such approach did work, yet one can hardly avoid to wonder if there is some profound reason for that to happen.

¹A recent overview of the status of the AdS-CFT correspondence is given in Ref.[4].

²For a review of Chern-Simons gravity see ref.[18].

An important clue is that the boundary term in ref.[19] can be understood as coming from the extension from Chern-Simons forms to transgression forms³, as mentioned in ref.[19] and discussed in detail in refs.[23, 24]. Transgression forms involve two gauge potentials A and \bar{A} , and Chern-Simons forms are just transgression forms with $\bar{A} = 0$.

The extension of Chern-Simons to transgression forms as a device to regularize the theory was done in 2+1 dimensions in ref.[25] (where the second field \bar{A} was understood as a fixed reference background), and in the context of actions for extended objects in refs.[26, 27] (here with both fields taken as dynamical). Afterwards, in the above mentioned refs.[19, 23, 24], it was shown how to use transgressions to properly regularize Chern-Simons gravity theories in arbitrary dimension. Other works using transgressions as actions are refs.[28, 29, 30, 31].

The key point in the transition from Chern-Simons to transgression forms is gauge invariance. While Chern-Simons forms are quasi-invariant, changing by a closed form under gauge transformations, transgression forms are truly gauge invariant. For instance the motivation for that transition in refs.[26, 27] was to have a truly gauge invariant action even in the case of branes with boundaries. Thus the results of refs.[19, 23, 24] can be construed as follows: the boundary terms dictated by the gauge principle turn out to be the ones that properly regularize the action.

The question that naturally arises is then: is there a symmetry principle which somehow explains why the boundary term of ref.[13] works? The answer is affirmative, and that is one of the main results of this paper.

By analogy with the case of the transgressions for the AdS group [24] we introduce an action for odd-dimensional AdS gravity with two sets of fields, that is that in addition to the vielbein e^a and the spin connection ω^{ab} with support in a manifold \mathcal{M} , we have a vielbein \bar{e}^a and a spin connection $\bar{\omega}^{ab}$ with support in a manifold $\bar{\mathcal{M}}$ with a common boundary with \mathcal{M} (that is $\partial\mathcal{M} \equiv \partial\bar{\mathcal{M}}$).

The complication of having an additional set of fields is compensated by:

- i. The arising of an enhanced 'almost off-shell' local AdS/Conformal symmetry for a very special choice of the boundary term.
- ii. The fact that both the 'background subtraction' and the 'kounterterms' approach can be regarded as particular cases of this framework. We show that the action principle of ref.[13] arises for a specific choice of the second field (regarded as a 'reference configuration' or 'vacuum'). The 'background subtraction' regularization corresponds to a different choice of the second field or 'reference configuration'. It is however very important to emphasize again that in the 'kounterterms' case there is no extra input of information required besides the original A configuration, as the \bar{A} is constructed from it in a direct way.

The Noether charges are constructed in general for the action given, and the charge of ref.[13] is a particular case of the general formula.

As computations in the kounterterms side of the present framework were

³For mathematical background on transgression forms see refs.[20, 21, 22].

done in ref.[13, 14, 15], here I do some 'background subtraction' computations as an application, and to show that the proposed method does indeed work. It is shown that for Schwarzschild-AdS black holes the charge associated to the time-like Killing vector is finite and is indeed the mass. The Euclidean action for Schwarzschild-AdS black holes with different horizon topologies is computed, with suitable backgrounds for each case, and it turns out to be finite, and to yield the right thermodynamics.

The important point of the relationship between the standard counterterms approach and our kounterterms approach has been discussed in refs.[14, 15] therefore I refer the reader to those papers on that regard.

2 The action

2.1 General setting

The reader be warned that the presentation of this section is somehow indirect. I first write the AdS transgression form derived in ref.[24], and the transgression action discussed there. Afterwards I consider an action for AdS gravity with a boundary term 'borrowed' from the AdS transgression with a coefficient to be determined, plus a doubling of the fields analogous to the one in the transgression. We will show that with the proper coefficient the resulting action has an enhanced symmetry and is properly regularized.

2.1.1 Review of AdS Transgressions

We briefly review in this sub-subsection some results from ref.[24] that we will use in what follows.

The AdS transgression in dimension $d = 2n + 1$ is⁴ [24]

$$\mathcal{T}_{2n+1} = \kappa \int_0^1 dt \epsilon (R + t^2 e^2)^n e - \kappa \int_0^1 dt \epsilon (\tilde{R} + t^2 \tilde{e}^2)^n \tilde{e} + d \alpha_{2n} \quad (1)$$

where

$$\alpha_{2n} = -\kappa n \int_0^1 dt \int_0^1 ds \epsilon \theta e_t \left\{ tR + (1-t)\tilde{R} - t(1-t)\theta^2 + s^2 e_t^2 \right\}^{n-1} \quad (2)$$

Here e^a and \tilde{e}^a are the two vielbeins and ω^{ab} and $\tilde{\omega}^{ab}$ the two spin connections, $R = d\omega + \omega^2$ and $\tilde{R} = d\tilde{\omega} + \tilde{\omega}^2$ are the corresponding curvatures, $\theta = \omega - \tilde{\omega}$ and

⁴I will use a compact notation where ϵ stands for the Levi-Civita symbol $\epsilon_{a_1 \dots a_d}$ and wedge products of differential forms are understood. For instance:

$$\epsilon R e^{d-2} \equiv \epsilon_{a_1 a_2 \dots a_d} R^{a_1 a_2} \wedge e^{a_3} \wedge \dots \wedge e^{a_{d-2}}$$

This clarification should be enough to follow what follows, but if there is any doubt on notation check [13, 19, 24].

$e_t = te + (1-t)\bar{e}$. Written in a more compact way

$$\alpha_{2n} = -\kappa n \int_0^1 dt \int_0^1 ds \epsilon \theta e_t \bar{R}_{st}^{n-1} \quad (3)$$

where

$$\bar{R}_{st} = tR + (1-t)\tilde{R} - t(1-t)\theta^2 + s^2 e_t^2$$

The action for transgressions for the AdS group is taken to be [24]

$$I_{Trans} = \kappa \int_{\mathcal{M}} \int_0^1 dt \epsilon (R + t^2 e^2)^n e - \kappa \int_{\bar{\mathcal{M}}} \int_0^1 dt \epsilon (\tilde{R} + t^2 \bar{e}^2)^n \bar{e} + \int_{\partial \mathcal{M}} \alpha_{2n} \quad (4)$$

where \mathcal{M} and $\bar{\mathcal{M}}$ are two manifolds with a common boundary, that is the boundaries of \mathcal{M} and $\bar{\mathcal{M}}$ coincide $\partial \mathcal{M} \equiv \partial \bar{\mathcal{M}}$. Notice that this is a generalization from the simpler case where $\mathcal{M} \equiv \bar{\mathcal{M}}$, which is physically motivated by the fact that both 'sheets' may not even have the same topology (for instance if \mathcal{M} is a black hole spacetime and $\bar{\mathcal{M}}$ is the AdS spacetime).

We write the transgression action as

$$I_{Trans} = \int_{\mathcal{M}} L_{LLCS} - \int_{\bar{\mathcal{M}}} \bar{L}_{LLCS} + \int_{\partial \mathcal{M}} \alpha_{2n} \quad (5)$$

where the Lanczos-Lovelock-Chern-Simons lagrangian L_{LLCS} is

$$L_{LLCS} = \kappa \int_0^1 dt \epsilon (R + t^2 e^2)^n e \quad (6)$$

The transgression form is invariant under gauge transformations for the AdS gauge group $SO(d-2,2)$. These are the Lorentz transformations

$$\delta \omega^{ab} = -d\lambda^{ab} - \omega_c^a \lambda^{cb} - \omega_c^b \lambda^{ac} \quad , \quad \delta e^a = \lambda^a_b e^b \quad (7)$$

and gauge translations

$$\delta \omega^{ab} = e^b \lambda^a - e^a \lambda^b \quad , \quad \delta e^a = -d\lambda^a - \omega_b^a \lambda^b \quad (8)$$

Writing the AdS gauge connection as⁵

$$A = \frac{\omega^{ab}}{2} J_{ab} + e^a P_a$$

⁵Actually as a gauge connection has dimensions of $(length)^{-1}$ we should write

$$A = \frac{\omega^{ab}}{2} J_{ab} + \frac{e^a}{l} P_a$$

where l is the 'AdS radius'. We choose $l = 1$ through all the present paper, as it is straightforward to reintroduce l everywhere using dimensional analysis.

where J_{ab} and P_a are the generators of the AdS group (for Lorentz transformations and translations respectively) and the gauge parameter

$$\lambda = \frac{\lambda^{ab}}{2} J_{ab} + \lambda^a P_a$$

the AdS gauge transformations take the compact form

$$\delta A = -D\lambda = -d\lambda - A\lambda + \lambda A$$

Invariance under Lorentz transformations is immediate, as all the ingredients of the action are Lorentz covariant and are contracted with the Levi-Civita invariant tensor ($\theta = \omega - \overline{\omega}$ is Lorentz covariant too, unlike ω or $\overline{\omega}$), but invariance under translations is very non trivial, yet true.

The transgression action is also invariant under the AdS gauge transformations, but it is only necessary that the transformations on \mathcal{M} and $\overline{\mathcal{M}}$ agree on $\partial\mathcal{M}$. The variation of the boundary term cancels the variations of both bulk terms.

2.1.2 Extended action for General Relativity

For General Relativity we consider the action

$$I_{GR} = \kappa \int_{\mathcal{M}} \epsilon \left[\frac{1}{d-2} R e^{d-2} + \frac{1}{d} e^d \right] - \kappa \int_{\overline{\mathcal{M}}} \epsilon \left[\frac{1}{d-2} \tilde{R} \tilde{e}^{d-2} + \frac{1}{d} \tilde{e}^d \right] + \chi_n \int_{\partial\mathcal{M}} \alpha_{2n} \quad (9)$$

with the same α_{2n} . We will see in several ways that the proper value for the constant χ_n is $\chi_n = \frac{1}{n(d-2)f(n-1)}$ with $f(n-1) = \int_0^1 dt (t^2 - 1)^{n-1}$. The action I_{GR} is explicitly invariant under Lorentz transformations, just like I_{Trans} . We write I_{GR} as

$$I_{GR} = \int_{\mathcal{M}} L_{EH} - \int_{\overline{\mathcal{M}}} \overline{L}_{EH} + \chi_n \int_{\partial\mathcal{M}} \alpha_{2n} \quad (10)$$

with the Einstein-Hilbert lagrangian L_{EH} given by

$$L_{EH} = \kappa \epsilon \left[\frac{1}{d-2} R e^{d-2} + \frac{1}{d} e^d \right] \quad (11)$$

Notice the doubling of the fields analogous to the transgression. In a saddle point evaluation of the euclidean action, or in the evaluation of the Noether charges the configuration A will be the one of interest while the configuration \overline{A} will be a reference 'vacuum', giving the 'background subtraction' or the 'counterterms' approach, depending on the nature of that 'vacuum'.

It is important however to emphasize that there is no 'little flag' labeling one of the configurations as the dynamical one and the other one as a background, and I believe in general both must be treated in the same footing. In particular

one could consider saddle point configurations with the roles of A and \bar{A} , namely the 'configuration of interest' and 'the vacuum', interchanged, considering the euclidean action with the opposite sign⁶.

2.2 Variation of the action and field equations

The variation of the transgression action yields

$$\begin{aligned} \delta I_{Trans} = & \kappa \int_{\mathcal{M}} \left[\epsilon \bar{R}^n \delta e + n \epsilon \bar{R}^{n-1} T \delta \omega \right] - \kappa \int_{\mathcal{M}} \left[\epsilon \bar{R}^n \delta \bar{e} + n \epsilon \bar{R}^{n-1} \bar{T} \delta \bar{\omega} \right] + \\ & + \int_{\partial \mathcal{M}} \left[-n \kappa \int_0^1 dt \epsilon (R + t^2 e^2)^{n-1} e \delta \omega + n \kappa \int_0^1 dt \epsilon (\tilde{R} + t^2 \bar{e}^2)^{n-1} \bar{e} \delta \bar{\omega} + \delta \alpha_{2n} \right] \end{aligned} \quad (12)$$

where $\bar{R} = R + e^2$ and $\bar{\tilde{R}} = \tilde{R} + \bar{e}^2$. From this variation we can read the transgression field equations [24]

$$\begin{aligned} \epsilon \bar{R}^n &= 0 \quad , \quad \epsilon \bar{R}^{n-1} T = 0 \\ \epsilon \bar{\tilde{R}}^n &= 0 \quad , \quad \epsilon \bar{\tilde{R}}^{n-1} \bar{T} = 0 \end{aligned} \quad (13)$$

For general relativity the variation of the action is

$$\begin{aligned} \delta I_{GR} = & \kappa \int_{\mathcal{M}} \left[\epsilon \bar{R} e^{d-3} \delta e + \epsilon T e^{d-3} \delta \omega \right] - \kappa \int_{\mathcal{M}} \left[\epsilon \bar{\tilde{R}} \bar{e}^{d-3} \delta \bar{e} + \epsilon \bar{T} \bar{e}^{d-3} \delta \bar{\omega} \right] + \\ & + \int_{\partial \mathcal{M}} \left[-\kappa \epsilon \frac{e^{d-2}}{d-2} \delta \omega + \kappa \epsilon \frac{\bar{e}^{d-2}}{d-2} \delta \bar{\omega} + \delta \alpha_{2n} \right] \end{aligned} \quad (14)$$

giving the standard field equations for General Relativity with a cosmological constant

$$\begin{aligned} \epsilon \bar{R} e^{d-3} &= 0 \quad , \quad \epsilon T e^{d-3} = 0 \\ \epsilon \bar{\tilde{R}} \bar{e}^{d-3} &= 0 \quad , \quad \epsilon \bar{T} \bar{e}^{d-3} = 0 \end{aligned} \quad (15)$$

If the vielbein is invertible the equation of motion $\epsilon T e^{d-3} = 0$ implies $T = 0$ (and $\epsilon \bar{T} \bar{e}^{d-3} = 0$ implies $\bar{T} = 0$).

It is of course clear that the equations of both theories are quite different and that solutions for one theory will not in general be solutions for the other.

⁶The idea is that the time variables of 'observers' in each configuration must be regarded as having opposite signs, so that each of them would see its 'piece' of the action as the positive one, and none would see undesirable effects as an exponentially enhanced probability of black hole nucleation from its past to its future (I am grateful to an anonymous referee for rising this question).

2.3 Boundary conditions

When we obtained the field equations in the previous section, we should have supplemented the action with suitable boundary conditions that make the boundary contribution to the variation of the action vanish, so that the action is truly an extremum when the field equations hold. Here we present two such conditions, though there may be others.

The first one, considered in refs.[13, 19], which we called *background independent configuration*, corresponds to the reference configuration \bar{A} chosen as

$$\bar{e} = 0, \quad \bar{\omega}^{ij} = \omega^{ij} \quad \text{and} \quad \bar{\omega}^{1j} = 0, \quad (16)$$

where 1 corresponds to the direction normal to the boundary (the normal being e^1) and i, j are different from 1. Then $\theta^{ij} = 0$ and $\theta^{1i} = \omega^{1i}$, $\bar{T} = 0$ and $\bar{R} = R - \theta^2$ for the components with support in the boundary. This configuration is the most natural and economical because given e and ω no further information is required, and the bulk term in $\bar{\mathcal{M}}$ vanishes, which can be interpreted in the sense that the second manifold is not even necessary. It is straightforward to check that for this configuration the General Relativity action eq.(9) reduces to the one discussed in ref.[13], where it was shown that it gives a well defined action principle for 'asymptotically locally AdS' (ALAdS) space-times.

The second one, discussed for transgressions in ref.[24], corresponds to

$$\Delta A \equiv A - \bar{A} \rightarrow 0$$

with a fast enough fall-off to kill the boundary term when the coordinate along the direction normal to the boundary approach the boundary. Looking at eq.(14) we see that in this case $\int_{\partial\mathcal{M}} \alpha_{2n} \rightarrow 0$ just as in ref.[24], while the remaining part of the boundary contribution to the variation

$$\int_{\partial\mathcal{M}} \left[-\kappa\epsilon \frac{e^{d-2}}{d-2} \delta\omega + \kappa\epsilon \frac{\bar{e}^{d-2}}{d-2} \delta\bar{\omega} \right]$$

will clearly vanish if $e - \bar{e} \rightarrow 0$ and $\omega - \bar{\omega} \rightarrow 0$ fast enough towards the boundary, as assumed. The fulfilment of this condition was explicitly checked for the configurations considered in the concrete examples below.

I find it tempting to name the second condition 'boundary without boundary' condition, as one may regard the manifolds \mathcal{M} and $\bar{\mathcal{M}}$ with opposite orientation (lets call it $\bar{\mathcal{M}}^*$) joined at $\partial\mathcal{M}$ as a single topological manifold $\mathcal{M} \cup \bar{\mathcal{M}}^*$, which however would not be a smooth manifold in general. If the boundary were at a finite distance boundary condition would mean that there is no discontinuity across the boundary. In fact for a boundary at a finite distance the fields would have no discontinuity and the boundary term would be zero, meaning the boundary is a fictitious one. I find the situation somewhat reminiscent of the 'method of images' in electrostatic, where a physical situation with a boundary is replaced by a configuration without a boundary in a wider region.

2.4 Enhanced symmetry: local AdS symmetry for General Relativity

As mentioned before both I_{Trans} and I_{GR} are explicitly invariant under Lorentz transformations. The transgression action is in addition invariant off-shell under gauge translations and hence under the whole AdS group. We will exploit that off-shell invariance to show that the GR action with the boundary term given and for a particular value of the coefficient χ_n is invariant under the AdS group when certain conditions are fulfilled.

We only need to consider gauge translations generated by the gauge parameter λ^a . The variation of I_{Trans} is in this case

$$\begin{aligned} \delta_\lambda I_{Trans} = & \int_{\partial\mathcal{M}} \{-\kappa\epsilon\bar{R}^n\lambda + \kappa\epsilon\bar{\tilde{R}}^n\lambda - \\ & + 2n\kappa \int_0^1 dt\epsilon(R+t^2e^2)^{n-1}e^2\lambda - 2n\kappa \int_0^1 dt\epsilon(\tilde{R}+t^2\bar{e}^2)^{n-1}\bar{e}^2\lambda + \delta_\lambda\alpha_{2n}\} \end{aligned} \quad (17)$$

But $\delta_\lambda I_{Trans} = 0$ owing to the off-shell gauge invariance of the transgression action, for any field configuration. Then

$$\begin{aligned} \int_{\partial\mathcal{M}} \delta_\lambda\alpha_{2n} = & \int_{\partial\mathcal{M}} \{+\kappa\epsilon\bar{R}^n\lambda - \kappa\epsilon\bar{\tilde{R}}^n\lambda + \\ & - 2n\kappa \int_0^1 dt\epsilon(R+t^2e^2)^{n-1}e^2\lambda + 2n\kappa \int_0^1 dt\epsilon(\tilde{R}+t^2\bar{e}^2)^{n-1}\bar{e}^2\lambda\} \end{aligned} \quad (18)$$

On the other hand the variation of the GR action under gauge translations is

$$\begin{aligned} \delta_\lambda I_{GR} = & \kappa \int_{\mathcal{M}} \epsilon(d-3)\bar{R}T e^{d-4}\lambda - \kappa \int_{\mathcal{M}} \epsilon(d-3)\bar{\tilde{R}}\bar{T}\bar{e}^{d-4}\lambda + \\ & + \int_{\partial\mathcal{M}} \{2\kappa\epsilon\frac{e^{d-1}}{d-2}\lambda - 2\kappa\epsilon\frac{\bar{e}^{d-1}}{d-2}\lambda - \kappa\epsilon\bar{R}e^{d-3}\lambda + \kappa\epsilon\bar{\tilde{R}}\bar{e}^{d-3}\lambda + \chi_n\delta_\lambda\alpha_{2n}\} \end{aligned} \quad (19)$$

where $\delta_\lambda\alpha_{2n}$ is given above.

We will ask for the following conditions:

(i) Vanishing of the bulk terms

$$\epsilon(d-3)\bar{R}T e^{d-4} = 0 \quad (20)$$

$$\epsilon(d-3)\bar{\tilde{R}}\bar{T}\bar{e}^{d-4} = 0 \quad (21)$$

This conditions are certainly fulfilled if the torsions vanish $T = \bar{T} = 0$, which seems like a natural and rather weak condition, but there may be other interesting configurations for which the the torsion does not vanish but the bulk terms are still zero, for instance some generalization to AdS gravity of the configurations studied by Chandía and Zanelli [32], having $\bar{R} = \bar{\tilde{R}} = 0$ but non zero torsions.

(ii) Asymptotic solution to GR equations of motion. We require that

$$\int_{\partial\mathcal{M}} \{\epsilon \bar{R} e^{d-3} \lambda - \epsilon \bar{\tilde{R}} \bar{e}^{d-3} \lambda\} = 0 \quad (22)$$

which is satisfied if the GR equations $\epsilon \bar{R} e^{d-3} = 0$ and $\epsilon \bar{\tilde{R}} \bar{e}^{d-3} = 0$ are satisfied asymptotically (not necessarily in all the bulk) with a fast enough fall off when approaching the boundary, but may also be satisfied with weaker conditions owing to cancellation between both terms.

(iii) Asymptotically AdS configurations. We require that the configuration are asymptotically AdS, $\bar{R} = 0$ and $\bar{\tilde{R}} = 0$, or $R = -e^2$ and $\tilde{R} = -\bar{e}^2$, with a fast enough fall off when approaching the boundary to make

$$\int_{\partial\mathcal{M}} \{\epsilon \bar{R}^n \lambda - \epsilon \bar{\tilde{R}}^n \lambda\} = 0$$

and to make

$$\begin{aligned} & \int_{\partial\mathcal{M}} \{-2n\kappa \int_0^1 dt \epsilon (R + t^2 e^2)^{n-1} e^2 \lambda + 2n\kappa \int_0^1 dt \epsilon (\tilde{R} + t^2 \bar{e}^2)^{n-1} \bar{e}^2 \lambda\} = \\ &= \int_{\partial\mathcal{M}} \{-2n\kappa \int_0^1 dt \epsilon (t^2 - 1)^{n-1} e^{d-1} \lambda + 2n\kappa \int_0^1 dt \epsilon (t^2 - 1)^{n-1} \bar{e}^{d-1} \lambda\} = \\ &= \int_{\partial\mathcal{M}} \{-2n\kappa f(n-1) \epsilon e^{d-1} \lambda + 2n\kappa f(n-1) \epsilon \bar{e}^{d-1} \lambda\} \quad (23) \end{aligned}$$

It is important to remark that condition (iii) does not imply condition (ii), even though the AdS configurations are solutions of the GR equations, because an asymptotic fall off fast enough to make zero the terms required in (iii) may not be fast enough to kill the terms required in (ii).

It turns out that if (i), (ii) and (iii) are satisfied and $\chi_n = \frac{1}{n(d-2)f(n-1)}$ then the GR action is invariant under gauge translations

$$\delta_\lambda I_{GR} = 0$$

which together with its Lorentz invariance implies that under this conditions I_{GR} is invariant under the full AdS group (which is the conformal group in $d-1$ dimensions).

Notice that the term

$$\int_{\partial\mathcal{M}} \left\{ 2\kappa \epsilon \frac{e^{d-1}}{d-2} \lambda - 2\kappa \epsilon \frac{\bar{e}^{d-1}}{d-2} \lambda \right\}$$

in $\delta_\lambda I_{GR}$ is not automatically zero, so there is a non trivial cancellation of this term with $\int_{\partial\mathcal{M}} \chi_n \delta_\lambda \alpha_{2n}$. It is this very non trivial cancellation what fixes the relative coefficient χ_n to the unique value found, and there lies the heart of the

invariance presented in this subsection.

A slightly different case is the above mentioned *background independent configuration*, considered in refs.[13, 19], where $\bar{\epsilon} = 0$, $\bar{\omega}^{ij} = \omega^{ij}$ and $\bar{\omega}^{1j} = 0$, where 1 corresponds to the direction normal to the boundary (the normal being e^1) and i, j are different from 1. Then $\theta^{ij} = 0$ and $\theta^{1i} = \omega^{1i}$, $\bar{T} = 0$ and $\tilde{R} = R - \theta^2$ for the components with support in the boundary.

For the *background independent configuration* conditions (i) and (ii) are the same, being even more easily fulfilled for $\bar{\epsilon}$ and $\bar{\omega}$. The condition (iii) is modified for $\bar{\epsilon}$ and $\bar{\omega}$ because $\bar{\tilde{R}} = \tilde{R}$, and we require for those fields

$$\int_{\partial\mathcal{M}} \{\tilde{R}^n \lambda\} = 0$$

That is enough to make

$$\delta_\lambda I_{GR} = 0$$

which implies that under this conditions I_{GR} is invariant under the full AdS group. It must be emphasized however that under generic AdS gauge transformations the 'gauge condition' $\bar{\epsilon} = 0$ is not preserved.

The previous considerations are valid for the euclidean GR action with periodic boundary conditions in the euclidean time, but also for the lorentzian GR action provided that the gauge parameters in the initial and final time space-like hypersurfaces vanish. The vanishing of the gauge parameters for the initial and final hypersurfaces in the lorentzian case is required because otherwise the boundary terms of the variation coming from those hypersurfaces would spoil the AdS invariance of the GR action.

In order to be more explicit about what it means a fast enough fall off in conditions (ii) and (iii) we can be more specific about the gauge parameter and afterwards look at the topological black hole solutions considered in the next sections [33, 34, 35] as examples to show that the class of configurations satisfying both conditions is not only not empty but rather quite wide.

We may consider a gauge parameter λ which goes at most as the radial coordinate r as $r \rightarrow \infty$, that is $\lambda = \mathcal{O}(r)$. For instance a parameter λ such that it is covariantly constant for AdS, $D\lambda = 0$ does satisfy that condition⁷. In fact a dependence of λ on a higher power of r will generate gauge transformations that are singular at the boundary.

In that case condition (ii) is satisfied if $\epsilon \bar{R} e^{d-3}$ and $\epsilon \bar{\tilde{R}} \bar{\epsilon}^{d-3}$ fall off asymptotically faster than $1/r$. The behaviour in the bulk is clearly not constrained by condition (ii).

⁷In that case, with the coordinates and notation of next section we should have $\lambda^1 = \lambda^{0m} = 0$, $\lambda^0 = \lambda^{01} = C^{(1)}r$, $\lambda^m = -\lambda^{0m} = C^{(m)}r$ and $\lambda_n^m \bar{\epsilon}^n = \omega_n^m C^{(m)}$, where the $C^{(a)}$'s are arbitrary constants.

Concerning condition (iii), to make

$$\int_{\partial\mathcal{M}} \{\epsilon\overline{R}^n \lambda - \epsilon\tilde{R}^n \lambda\} = 0$$

⁸it will be necessary that $\epsilon\overline{R}^n$ and $\epsilon\tilde{R}^n$ would fall off faster than $1/r$. That is a quite weak condition for asymptotically locally AdS space-times, easily met by the black hole solutions considered in the next sections, for which the components of \overline{R} and \tilde{R} with support at the boundary scale with r as $\mathcal{O}(1/r^{d-3})$ (everything else in the integral being just 'angular factors').

Concerning the validity of eq.(23) what we need is that asymptotically $\overline{R} = 0$ and $\tilde{R} = 0$ with a fast enough fall off to allow us to replace R by $-e^2$ and \tilde{R} by $-\tilde{e}^2$ in the integrals. The leading order of the components with support at the boundary of e^2 and \tilde{e}^2 go as r^2 for AdS asymptotics, and we have $n - 1 = \frac{d-3}{2}$ factors $(R + t^2 e^2)$ and a factor e^2 (or $(\tilde{R} + t^2 \tilde{e}^2)$ and a factor \tilde{e}^2). We can write $R = \overline{R} - e^2$ ($\tilde{R} = \tilde{\overline{R}} - \tilde{e}^2$), then if we suppose $\lambda = \mathcal{O}(r)$ as before, the term with one \overline{R} and $d - 3$ e 's vanishes because of (ii) above (and the same holds for the 'tilde' fields for all the paragraph) then the next order corresponds to two \overline{R} and $d - 5$ e 's, to kill which we must require that the components of \overline{R} and $\tilde{\overline{R}}$ with support at the boundary fall off with r faster than $1/r^{\frac{d-4}{2}}$.

As with condition (ii), condition (iii) does not impose any restriction on the behaviour on the bulk.

If we consider as an example the black hole solutions of the next sections, there is no problem with condition (ii), as those configurations actually satisfy the field equations $\epsilon\overline{R}e^{d-3} = 0$ and $\epsilon\tilde{R}\tilde{e}^{d-3} = 0$, that is far more than what we need to require.

The first part of condition (iii) is also verified, as it was just mentioned. The second part of condition (iii) is also verified because as we already said the components of \overline{R} and $\tilde{\overline{R}}$ with support at the boundary scale with r as $\mathcal{O}(1/r^{d-3})$.

For the *background independent configuration* conditions (i) and (ii) lead to the same requirements again. The condition (iii) which is modified for \bar{e} and \bar{w} to

$$\int_{\partial\mathcal{M}} \epsilon\{\tilde{R}^n \lambda\} = 0$$

leads to a required fall off faster than $1/r$ for $\epsilon\{\tilde{R}^n$ if $\lambda = \mathcal{O}(r)$, which is satisfied for the above mentioned black hole configurations.

⁸Notice that in these expressions $n = \frac{d-1}{2}$.

3 Euclidean action and thermodynamics for Schwarzschild black holes.

In this section I will evaluate the euclidean action for Schwarzschild black holes with different asymptotic topologies, with suitable reference backgrounds. To that end I will first evaluate the euclidean action with A and \bar{A} taken to be two black hole like configurations with the same asymptotic topology, and eventually chose the the right \bar{A} so that for a given black hole configuration A the whole euclidean geometry is non singular.

I will consider the action of eq.(9) and the black hole solutions of refs.[33, 34, 35]. This solutions have line element

$$ds^2 = -\Delta^2(r)dt^2 + \frac{dr^2}{\Delta^2(r)} + r^2 d\Sigma_{d-2}^2 \quad (24)$$

with

$$\Delta^2 = \gamma - \frac{2GM}{r^{d-3}} + r^2 \quad (25)$$

where $d\Sigma_{d-2}^2$ is the line element of the $(d-2)$ -dimensional manifold of constant curvature proportional to $\gamma = 1, 0, -1$. The event horizon r_+ is given by $\Delta(r_+) = 0$.

3.1 Evaluation of the Euclidean Action

In order to evaluate the euclidean action for two black hole configurations with masses M and \bar{M} respectively the relevant non vanishing ingredients are

$$\begin{aligned} e^0 &= \Delta dt \quad , \quad e^1 = \frac{1}{\Delta} dr \quad , \quad e^m = r \tilde{e}^m \\ \omega^{01} &= \left(\frac{\Delta^2}{2} \right)' dt \quad , \quad \omega^{1m} = -\Delta \tilde{e}^m \quad , \quad \omega^{mn} = \tilde{\omega}^{mn} \\ R^{01} &= - \left(\frac{\Delta^2}{2} \right)'' dt dr \quad , \quad R^{0m} = -\Delta \left(\frac{\Delta^2}{2} \right)' dt \tilde{e}^m \\ R^{1m} &= -\frac{1}{\Delta} \left(\frac{\Delta^2}{2} \right)' dr \tilde{e}^m \quad , \quad R^{mn} = (\gamma - \Delta^2) \tilde{e}^m \tilde{e}^n \end{aligned} \quad (26)$$

with

$$\Delta^2 = \gamma - \frac{2GM}{r^{d-3}} + r^2$$

(hence $\left(\frac{\Delta^2}{2} \right)' = r + (d-3)\frac{GM}{r^{d-2}}$), for $\gamma = 1, 0, -1$. Similar expressions hold with $e \rightarrow \bar{e}$, $\omega \rightarrow \bar{\omega}$, $R \rightarrow \bar{R}$, $\Delta \rightarrow \bar{\Delta}$ and $M \rightarrow \bar{M}$. We then have

$$\theta^{mn} = -(\Delta - \bar{\Delta}) \tilde{e}^m \quad , \quad (\theta^2)^{mn} = -(\Delta - \bar{\Delta}) \tilde{e}^m \tilde{e}^n$$

$$\begin{aligned}\theta^{01} &= \left[\left(\frac{\Delta^2}{2} \right)' - \left(\frac{\bar{\Delta}^2}{2} \right)' \right] dt, \quad (\theta^2)^{0m} = -(\Delta - \bar{\Delta}) \left[\left(\frac{\Delta^2}{2} \right)' - \left(\frac{\bar{\Delta}^2}{2} \right)' \right] dt \tilde{e}^m \\ e_t^0 &= [t\Delta + (1-t)\bar{\Delta}] dt, \quad e_t^m = r \tilde{e}^m \\ (e_t^2)^{0m} &= r(t\Delta + (1-t)\bar{\Delta}) dt \tilde{e}^m, \quad (e_t^2)^{mn} = r^2 \tilde{e}^m \tilde{e}^n\end{aligned}\quad (27)$$

We will need the components of $\bar{R}_{st} = tR + (1-t)\tilde{R} - t(1-t)\theta^2 + s^2 e_t^2$ with group indices mn and $0m$. Those are

$$\begin{aligned}(\bar{R}_{st})^{mn} &= \{ \gamma - [t\Delta + (1-t)\bar{\Delta}]^2 + s^2 r^2 \} \tilde{e}^m \tilde{e}^n \\ (\bar{R}_{st})^{0m} &= \left\{ -t\Delta \left(\frac{\Delta^2}{2} \right)' - (1-t)\bar{\Delta} \left(\frac{\bar{\Delta}^2}{2} \right)' + r(t\Delta + (1-t)\bar{\Delta}) s^2 \right\} dt \tilde{e}^m\end{aligned}\quad (28)$$

The bulk contribution to the euclidean action can be evaluated using the equations of motion $R + e^2 = 0$ and it is

$$I_E^{bulk} = 2\kappa(d-3)! \beta \Sigma_{d-2} [r_+^{2n}] - 2\kappa(d-3)! \beta \Sigma_{d-2} [\bar{r}_+^{2n}] \quad (29)$$

where Σ_{d-2} is the volume of the constant curvature manifold corresponding to the sections of fixed unity radius and fixed euclidean time, which in the spherically symmetric case we will also call Ω_{d-2} , coming from integration over the angular variables.

The boundary term is

$$\begin{aligned}\alpha_{2n} &= -\kappa n \int_0^1 dt \int_0^1 ds \{ 2\epsilon_{1m_1 0 m_2 \dots m_{2n-1}} \theta^{1m_1} e_t^0 \bar{R}_{st}^{m_2 m_3} \dots \bar{R}_{st}^{m_{2n-2} m_{2n-1}} + \\ &\quad + 4(n-1) \epsilon_{1m_1 m_2 0 m_3 \dots m_{2n-1}} \theta^{1m_1} e_t^{m_2} \bar{R}_{st}^{0 m_3} \bar{R}_{st}^{m_4 m_5} \dots \bar{R}_{st}^{m_{2n-2} m_{2n-1}} + \\ &\quad + 2\epsilon_{01 m_1 m_2 \dots m_{2n-1}} \theta^{01} e_t^{m_1} \bar{R}_{st}^{m_2 m_3} \dots \bar{R}_{st}^{m_{2n-2} m_{2n-1}} \} \quad (30)\end{aligned}$$

Inserting the expressions for the terms of this equation and taking in account signs coming from bringing the ϵ to its standard order, commuting differentials and an additional sign coming from the orientation of the boundary (or equivalently bringing the differential dr to the front from the canonical order $dt dr \tilde{e}^{m_1} \dots \tilde{e}^{m_{2n-1}} \rightarrow -dr dt \tilde{e}^{m_1} \dots \tilde{e}^{m_{2n-1}}$) we get

$$\begin{aligned}\int_{\partial \mathcal{M}} \alpha_{2n} &= \kappa n 2 \beta (d-2)! \Sigma_{d-2} \int_0^1 dt \int_0^1 ds \{ (\Delta - \bar{\Delta}) [t\Delta + (1-t)\bar{\Delta}] \times \\ &\quad \times [\gamma - (t\Delta + (1-t)\bar{\Delta})^2 + s^2 r^2]^{n-1} + \\ &\quad + 2(n-1) r (\Delta - \bar{\Delta}) [-t\Delta \left(\frac{\Delta^2}{2} \right)' - (1-t)\bar{\Delta} \left(\frac{\bar{\Delta}^2}{2} \right)' + r(t\Delta + (1-t)\bar{\Delta}) s^2] \times \\ &\quad \times [\gamma - (t\Delta + (1-t)\bar{\Delta})^2 + s^2 r^2]^{n-2} + \\ &\quad + \left[\left(\frac{\Delta^2}{2} \right)' - \left(\frac{\bar{\Delta}^2}{2} \right)' \right] r [\gamma - (t\Delta + (1-t)\bar{\Delta})^2 + s^2 r^2]^{n-1} \} \quad (31)\end{aligned}$$

We will drop terms that give a vanishing contribution when $r \rightarrow \infty$ and keep only divergent or finite contributions in that limit. To that end we notice that

$$\left(\frac{\Delta^2}{2}\right)' = r + (d-3)\frac{GM}{r^{d-2}}$$

and that for $r \rightarrow \infty$ we get

$$\Delta \rightarrow r + \frac{\gamma}{2r} - \frac{GM}{r^{d-2}}$$

and hence

$$\begin{aligned} \left(\frac{\Delta^2}{2}\right)' - \left(\frac{\overline{\Delta}^2}{2}\right)' &= (d-3)\frac{G(M-\overline{M})}{r^{d-2}} \quad , \quad \Delta - \overline{\Delta} \rightarrow -\frac{G(M-\overline{M})}{r^{d-2}} \\ \gamma - (t\Delta + (1-t)\overline{\Delta})^2 + s^2r^2 &\rightarrow (s^2-1)r^2 + \mathcal{O}(r) \\ (\Delta - \overline{\Delta}) \left[\left(\frac{\Delta^2}{2}\right)' - \left(\frac{\overline{\Delta}^2}{2}\right)' \right] &\rightarrow \mathcal{O}(r^{-3(d-2)}) \\ r(\Delta - \overline{\Delta})\Delta \left(\frac{\Delta^2}{2}\right)' &\rightarrow r^2(\Delta - \overline{\Delta})\Delta + \mathcal{O}(r^{-3(d-2)}) \\ r(\Delta - \overline{\Delta})\overline{\Delta} \left(\frac{\overline{\Delta}^2}{2}\right)' &\rightarrow r^2(\Delta - \overline{\Delta})\overline{\Delta} + \mathcal{O}(r^{-3(d-2)}) \end{aligned} \quad (32)$$

The boundary term is then

$$\begin{aligned} \int_{\partial\mathcal{M}} \alpha_{2n} &= \kappa n 2\beta(d-2)! \Sigma_{d-2} \int_0^1 ds \{ \int_0^1 dt (\Delta - \overline{\Delta}) [t\Delta + (1-t)\overline{\Delta}] \times \\ &\quad \times [\gamma - (t\Delta + (1-t)\overline{\Delta})^2 + s^2r^2]^{n-1} + \\ &\quad 2(n-1)r^2(s^2-1)(\Delta - \overline{\Delta})(t\Delta + (1-t)\overline{\Delta}) \times \\ &\quad \times [\gamma - (t\Delta + (1-t)\overline{\Delta})^2 + s^2r^2]^{n-2} \} + \\ &\quad + \kappa n 2\beta(d-2)! \Sigma_{d-2} \int_0^1 ds (d-3)G(M-\overline{M})(s^2-1)^{n-1} \end{aligned} \quad (33)$$

The integral in the parameter t can be done trough the substitution

$$u = \gamma - (t\Delta + (1-t)\overline{\Delta})^2 + s^2r^2$$

and the result is

$$\begin{aligned} \int_{\partial\mathcal{M}} \alpha_{2n} &= -\kappa\beta(d-2)! \Sigma_{d-2} \int_0^1 ds [u^n + 2nr^2(s^2-1)u^{n-1}] \Big|_{\gamma-\overline{\Delta}^2+s^2r^2}^{\gamma-\Delta^2+s^2r^2} + \\ &\quad + \kappa n 2\beta(d-2)! \Sigma_{d-2} \int_0^1 ds (d-3)G(M-\overline{M})(s^2-1)^{n-1} \end{aligned} \quad (34)$$

Notice that $\gamma - \Delta^2 + s^2 r^2 = (s^2 - 1)r^2 + \frac{2GM}{r^{d-3}}$ and $1 - \overline{\Delta}^2 + s^2 r^2 = (s^2 - 1)r^2 + \frac{2G\overline{M}}{r^{d-3}}$ and that α_{2n} is evaluated at the boundary where $r \rightarrow \infty$. Keeping only terms that will give a divergent or finite contribution we can expand

$$\begin{aligned} \left[(s^2 - 1)r^2 + \frac{2GM}{r^{d-3}} \right]^n &= (s^2 - 1)^n r^{2n} + n 2GM (s^2 - 1)^{n-1} + \dots \\ \left[(s^2 - 1)r^2 + \frac{2GM}{r^{d-3}} \right]^{n-1} &= (s^2 - 1)^{n-1} r^{2n-2} + (n-1) \frac{2GM}{r^2} (s^2 - 1)^{n-2} + \dots \end{aligned} \quad (35)$$

The divergent contributions cancel between the upper and lower limits of the integrals. The resulting α_{2n} is then

$$\int_{\partial\mathcal{M}} \alpha_{2n} = -\kappa \beta (d-2)! \Sigma_{d-2} f(n-1) n 2G(M - \overline{M}) \quad (36)$$

where

$$f(n-1) = \int_0^1 ds (s^2 - 1)^{n-1} = (-1)^{n-1} \frac{(n-1)! 2^{n-1}}{(2n-1)!!}$$

With the choice $\chi_n = \frac{1}{nf(n-1)(d-2)}$ the total action reads

$$I_E^{Total} = 2\kappa(d-3)! \beta \Sigma_{d-2} \{ [r_+^{2n}] - [\overline{r}_+^{2n}] - G(M - \overline{M}) \} \quad (37)$$

We can replace $\kappa = \frac{1}{2G(d-2)!\Omega_{d-2}}$ to get

$$I_E^{Total} = \beta \frac{\Sigma_{d-2}}{\Omega_{d-2}} \frac{1}{(d-2)G} \{ [r_+^{d-1}] - [\overline{r}_+^{d-1}] \} - \beta \frac{\Sigma_{d-2}}{\Omega_{d-2}} \frac{(M - \overline{M})}{(d-2)} \quad (38)$$

3.2 Determination of β and a suitable reference background

The euclidean time period β is determined by requiring that the euclidean solution be non singular. For generic black hole metrics of the form

$$ds^2 = \Delta^2(r) dt^2 + \frac{dr^2}{\Delta^2(r)} + r^2 d\Sigma_{d-2}^2$$

with the an event horizon r_+ given by $\Delta(r_+) = 0$ it turns out that

$$\beta = \frac{4\pi}{(\Delta^2)'(r_+)}$$

For a General Relativity black hole of mass M it is

$$\beta = \frac{4\pi}{(d-1)r_+ + \frac{(d-3)\gamma}{r_+}}$$

This implies

$$M = \frac{r_+^{d-3}}{2G}(\gamma + r_+^2)$$

In order to have a sensible thermodynamics the background configuration must be chosen as having an arbitrary $\bar{\beta}$. For $\gamma = 0$ and $\gamma = 1$ the proper configurations are the zero mass black holes, with $\bar{M} = 0$ and $\bar{r}_+ = 0$, which correspond to AdS for $\gamma = 1$. This configurations have $\bar{\beta} = \infty$, what amounts to an ill-defined or arbitrary $\bar{\beta}$, as it can be checked that this euclidean configurations are non singular for any $\bar{\beta}$. In particular, for the evaluation of the euclidean action of the previous section to make sense we must take $\bar{\beta} = \beta$.

In the case $\gamma = -1$ the situation is a little more complicated. Requiring an ill-defined $\bar{\beta} = \infty$ yields in this case

$$\bar{r}_+ = \pm \sqrt{\frac{d-3}{d-1}}$$

and as r is positive we must pick the positive root. This value of \bar{r}_+ gives

$$\bar{M} = -\frac{1}{(d-1)G} \left[\frac{d-3}{d-1} \right]^{\frac{d-3}{2}} \equiv M_0$$

Notice that M_0 is negative. Again for the evaluation of the euclidean action to make sense we need to take $\bar{\beta} = \beta$.

3.3 Black hole thermodynamics

We then get

$$I_E^{Total} = \beta \frac{\Sigma_{d-2}}{\Omega_{d-2}} \frac{1}{(d-2)G} [r_+^{d-1}] - \beta \frac{\Sigma_{d-2}}{\Omega_{d-2}} \frac{M}{(d-2)} + \beta \frac{\Sigma_{d-2}}{\Omega_{d-2}} M_0 \delta_{-1,\gamma} \quad (39)$$

which coincides with the result obtained in [13], except for a different M independent term (the one that yields the vacuum energy) in that case.

To discuss the black hole thermodynamics we use that the euclidean action I is related with the free energy F as $I = -\beta F$, while the free energy is related to the energy E and the entropy S as $F = E - TS = E - S/\beta$. Equivalently

$$I = -\beta E + S \quad (40)$$

Hence

$$E = -\frac{\partial I}{\partial \beta} = -\frac{\frac{\partial I}{\partial r_+}}{\frac{\partial \beta}{\partial r_+}} \quad (41)$$

The result of this calculation is

$$E = \frac{\Sigma_{d-2}}{\Omega_{d-2}} (M - M_0 \delta_{-1,\gamma}) \quad (42)$$

The entropy can be calculated from

$$S = I + \beta E \quad (43)$$

and it is

$$S = \frac{2\pi r_+^{2n-1}}{(2n-1)G} \frac{\Sigma_{d-2}}{\Omega_{d-2}} = \frac{r_+^{2n-1} \Sigma_{d-2}}{4G_N} = \frac{A}{4G_N} \quad (44)$$

in agreement with the Bekenstein-Hawking entropy. We used that the standard Newton constant G_N and the constant G are related as [35]

$$G = \frac{8\pi}{(d-2)\Omega_{d-2}} G_N$$

3.4 Discussion: boundary terms versus the Hawking-Page approach

As it was already pointed out in ref.[24] the background subtraction procedure used here is not the same as the one proposed by Gibbons and Hawking [5], or by Hawking and Page [6]. In those papers, the actions for two different configurations (for instance, for a black hole and Minkowski or AdS space) are subtracted, with the additional condition that the metrics match at a very large finite radius r_0 (eventually taken to infinity). In that case two different euclidean time intervals β and $\bar{\beta}$ are involved, because the condition

$$ds^2|_{r_0} = \bar{ds}^2|_{r_0}$$

implies

$$\Delta(r_0)\beta(r_0) = \bar{\Delta}(r_0)\bar{\beta}(r_0)$$

then, even though $\bar{\beta} \rightarrow \beta$ when $r_0 \rightarrow \infty$, there is an extra contribution to the total bulk action (the difference of the bulk actions for the configuration of interest and the 'background') coming from the difference of the β 's [6, 3].

In our approach there is always only one β , as it must be in order to integrate the boundary term B_{2n} , where both sets of vielbein and spin connections appear entangled, but we do have an extra contribution coming from that boundary term.

It is worthwhile to emphasize that boundary term contributions are absent in the Hawking-Page approach to asymptotically AdS space-times, as the Gibbons-Hawking term is zero in that case.

It is instructive to compare both methods for the concrete example of the Schwarzschild-AdS black hole with spherical symmetry, as the extra contribution coming in the Hawking-Page method from the differing β 's comes in our approach from the boundary term, and both methods agree in that case.

When it comes to the conserved charges, discussed in the next section, we will see for the concrete example of the Schwarzschild black hole that the boundary

term contribution is necessary to obtain a result for the mass in agreement with the one obtained from the thermodynamics. See in particular Section 4.2 below and the comment in the last paragraph of that section. It is hard to see where could this contribution come from in the Hawking-Page approach, as the Noether charges are defined as integrals in the boundary of spatial sections, therefore no integral in the euclidean time is done and β or $\bar{\beta}$ are not involved in the result.

4 Conserved charges from Noether's theorem

4.1 Noether's charges

The action is

$$I = \kappa \int_{\mathcal{M}} \epsilon \left[\frac{1}{d-2} R e^{d-2} + \frac{1}{d} e^d \right] - \kappa \int_{\mathcal{M}} \left[\frac{1}{d-2} \tilde{R} \tilde{e}^{d-2} + \frac{1}{d} \tilde{e}^d \right] + \chi_n \int_{\partial \mathcal{M}} \alpha_{2n} \quad (45)$$

where

$$\alpha_{2n} = -\kappa n \int_0^1 dt \int_0^1 ds \, \epsilon \theta e_t \left\{ tR + (1-t)\tilde{R} - t(1-t)\theta^2 + s^2 e_t^2 \right\}^{n-1} \quad (46)$$

χ_n is a constant relative factor and the boundaries coincide $\partial \mathcal{M} \equiv \partial \overline{\mathcal{M}}$. Applying Noether's theorem to this action we get the conserved current associated to the invariance under diffeomorphisms generated by the vector field ξ

$$\star j = dQ_\xi \quad (47)$$

with⁹

$$Q_\xi = \frac{\kappa}{(d-2)} \epsilon [e^{d-2} I_\xi \omega - \tilde{e}^{d-2} I_\xi \tilde{\omega}] + \chi_n I_\xi \alpha_{2n} \quad (48)$$

which is to be integrated at the spatial boundary $\partial \mathcal{S}$, which for instance for topological black holes is Σ^{d-2} . Here

$$I_\xi \alpha_{2n} = -\kappa n \epsilon \left\{ \int_0^1 dt \int_0^1 ds \, I_\xi \theta e_t \overline{R}_{st}^{n-1} - \int_0^1 dt \int_0^1 ds \, \theta I_\xi e_t \overline{R}_{st}^{n-1} + (n-1) \int_0^1 dt \int_0^1 ds \, \theta e_t I_\xi \overline{R}_{st} \overline{R}_{st}^{n-2} \right\} \quad (49)$$

where $\overline{R}_{st} = tR + (1-t)\tilde{R} - t(1-t)\theta^2 + s^2 e_t^2$.

⁹The contraction operator I_ξ is defined by acting on a p-form α_p as

$$I_\xi \alpha_p = \frac{1}{(p-1)!} \xi^\nu \alpha_{\nu \mu_1 \dots \mu_{p-1}} dx^{\mu_1} \dots dx^{\mu_{p-1}}$$

and being and anti-derivative in the sense that acting on the wedge product of differential forms α_p and β_q of order p and q respectively gives $I_\xi(\alpha_p \beta_q) = I_\xi \alpha_p \beta_q + (-1)^p \alpha_p I_\xi \beta_q$.

4.2 Black hole mass

We will evaluate the charge corresponding to $\xi = \frac{\partial}{\partial t}$ for two black hole configurations with masses M and \overline{M} respectively the relevant non vanishing ingredients are the same used in the evaluation of the euclidean action for black holes. We have

$$\int_{\partial S} \frac{\kappa}{(d-2)} \epsilon [e^{d-2} I_\xi \omega - \bar{e}^{d-2} I_\xi \bar{\omega}] = \frac{\kappa}{(d-2)} \Sigma_{d-2} (d-2)! r^{d-2} \left[\left(\frac{\Delta^2}{2} \right)' - \left(\frac{\overline{\Delta}^2}{2} \right)' \right] \quad (50)$$

At the boundary $r \rightarrow \infty$ we get

$$\int_{\partial S} \frac{\kappa}{(d-2)} \epsilon [e^{d-2} I_\xi \omega - \bar{e}^{d-2} I_\xi \bar{\omega}] = \frac{(d-3)}{(d-2)} \kappa \Sigma_{d-2} (d-2)! 2G(M - \overline{M}) \quad (51)$$

Furthermore we have the following non vanishing contribution to $I_\xi \alpha_{2n}$

$$\begin{aligned} I_\xi \alpha_{2n} = & -\kappa n \int_0^1 dt \int_0^1 ds \{ 2\epsilon_{01m_1 \dots m_{2n-1}} I_\xi \theta^{01} e_t^{m_1} \overline{R}_{st}^{m_2 m_3} \dots \overline{R}_{st}^{m_{2n-2} m_{2n-1}} - \\ & - 2\epsilon_{1m_1 0 m_2 m_3 \dots m_{2n-1}} \theta^{1m_1} I_\xi e_t^0 \overline{R}_{st}^{m_2 m_3} \dots \overline{R}_{st}^{m_{2n-2} m_{2n-1}} + \\ & + 4(n-1) \epsilon_{1m_1 m_2 0 m_3 \dots m_{2n-1}} \theta^{1m_1} e_t^{m_2} I_\xi \overline{R}_{st}^{0 m_3} \overline{R}_{st}^{m_4 m_5} \dots \overline{R}_{st}^{m_{2n-2} m_{2n-1}} \} \quad (52) \end{aligned}$$

Inserting the expressions for the terms of this equation and taking in account signs coming from bringing the ϵ to its standard order we get

$$\begin{aligned} \int_{\partial S} I_\xi \alpha_{2n} = & -2\kappa n (d-2)! \Sigma_{d-2} \int_0^1 dt \int_0^1 ds \left\{ \left[\left(\frac{\Delta^2}{2} \right)' - \left(\frac{\overline{\Delta}^2}{2} \right)' \right] \times \right. \\ & \times r [\gamma - (t\Delta + (1-t)\overline{\Delta})^2 + s^2 r^2]^{n-1} + \\ & + (\Delta - \overline{\Delta}) [t\Delta + (1-t)\overline{\Delta}] [\gamma - (t\Delta + (1-t)\overline{\Delta})^2 + s^2 r^2]^{n-1} + \\ & + 2(n-1)r(\Delta - \overline{\Delta}) \left[-t\Delta \left(\frac{\Delta^2}{2} \right)' - (1-t)\overline{\Delta} \left(\frac{\overline{\Delta}^2}{2} \right)' + r(t\Delta + (1-t)\overline{\Delta}) s^2 \right] \times \\ & \left. \times [\gamma - (t\Delta + (1-t)\overline{\Delta})^2 + s^2 r^2]^{n-2} \right\} \quad (53) \end{aligned}$$

Notice that the integrals in s and t are just the same we did in the evaluation of the euclidean action, then we can directly write down the result

$$\int_{\partial S} I_\xi \alpha_{2n} = \kappa (d-2)! \Sigma_{d-2} f(n-1) n 2G(M - \overline{M}) \quad (54)$$

where $f(n-1) = \int_0^1 ds (s^2 - 1)^{n-1} = (-1)^{n-1} \frac{(n-1)! 2^{n-1}}{(2n-1)!!}$. With the choice $\chi_n = \frac{1}{nf(n-1)(d-2)}$ the total charge reads

$$\int_{\partial S} Q_\xi = \kappa \Sigma_{d-2} (d-2)! 2G(M - \overline{M}) \quad (55)$$

We can replace $\kappa = \frac{1}{2G(d-2)!\Omega_{d-2}}$ to get

$$\int_{\partial S} Q_\xi = \frac{\Sigma_{d-2}}{\Omega_{d-2}}(M - \overline{M}) \quad (56)$$

which is the expected result. Notice that without the $I_\xi \alpha_{2n}$ contribution the charge would be

$$\int_{\partial S} Q_\xi^{bulk} = \frac{(d-3)}{(d-2)} \frac{\Sigma_{d-2}}{\Omega_{d-2}}(M - \overline{M}) \quad (57)$$

It is worthwhile to emphasize the significance of this result: the contribution to the Noether's conserved charge coming from the boundary term is **required** to get a value of the mass in agreement with the one coming from the thermodynamics.

5 Discussion and Conclusions

The results of this paper show that the boundary term suitable for properly regularizing odd-dimensional AdS gravity may be regarded as dictated or suggested by a symmetry principle, which could be seen as the reason for that particular term to work. The symmetry principle invoked is invariance under local transformations of the AdS group, and it holds 'almost off-shell'.

The calculations of the thermodynamics and Noether charges of AdS-Schwarzschild black holes with different topologies are new not in the results, but are quite different in the methods used, as the contribution of the boundary terms is crucial in our approach, as discussed in section 3.4 and the last paragraph of section 4.2. Our method provides a regularization procedure which allows those calculations to be made in a uniform and systematic way for any solution.

The present work rises several questions and could be extended in several directions:

The fact that the action considered, with the precise boundary term chosen, has that extra symmetry suggest that it may be relevant in the study of the AdS-CFT correspondence, as the AdS group in dimension d is the conformal group in dimension $d - 1$ ¹⁰. It is clear however that the possible relevance of the setup considered here to the AdS-CFT correspondence, beyond the remarks of ref.[13], is just an interesting open question, which would require to address several issues, such as studying the action presented here for Dirichlet boundary

¹⁰If that were the case, it would however be puzzling that the conformal symmetry that would be induced if one chooses boundary conditions at infinity that do not break the symmetry and integrates out the bulk degrees of freedom would have a local symmetry with the conformal group as gauge group, while the CFT side of the AdS-CFT correspondence involves a globally symmetric conformal field theory. It may be that integrating out the bulk degrees of freedom corresponding to A while keeping the degrees of freedom associated to the configuration \overline{A} of ref.[13, 19] as boundary degrees of freedom of the effective theory would reduce the symmetry from a local gauge redundancy to a global symmetry

conditions. Concrete calculations that could shed light on this matter could be done concerning the issue of the conformal anomaly of the boundary CFT, in the spirit of refs.[7, 10, 11, 12]. One could consider the situation in which \bar{A} for AdS gravity corresponds to the configuration used in our kounterterms approach, and read the conformal anomaly from the variation of the action under radial diffeos (which induce Weyl transformations in the boundary for the boundary conditions of ref.[19]) with \bar{A} kept fixed. This would correspond to the picture of the anomaly as arising from the non invariance of the regulator (or subtraction procedure), as \bar{A} in the our kounterterms approach could be seen as a regulator.

One direction that seems both interesting and accessible has to do with the extension to the supersymmetric case. It seems reasonable to guess that the boundary terms that would result from the transgression forms for suitable supersymmetric extensions of the AdS groups would regulate, with proper coefficients relative to the bulk, different versions of standard supergravities which are extensions of AdS gravities. In this respect it is worthwhile to remark that the issue of picking the right boundary terms in standard supergravities related to the M-theory is an important one, even with possible implications for phenomenology, as it can be seen in the recent papers by Moss [36] and references therein. The procedure followed in ref.[36] was to compute the boundary terms order by order in an expansion in powers of some parameter, which is not guaranteed to end.

It is also possible and worth exploring that the transgression action of eq.(4) would be even better suited for eventual application to the AdS-CFT correspondence, as its invariance under gauge transformations for the AdS/conformal group is exact (and of course off-shell) without further requirements or conditions. That would not be surprising if one believes that the effective field theory description of the M-theory with corrections of higher order in the curvature included could in fact be a Chern-Simons supergravity, which was originally proposed in ref. [37] and also explored in refs.[38, 39, 40].

Finally, the action has an obvious symmetry under the interchange of A and \bar{A} and change of sign of the action, which may have non trivial consequences worth exploring. It is intriguing that Linde [41, 42] studied a model of gravity coupled with scalar fields where the field content (including gravity) is duplicated (as it is for us) and a similar symmetry, which Linde calls *antipodal symmetry*, is the key to a way to solve the cosmological constant problem¹¹.

Acknowledgments

¹¹In ref.[41, 42] both fields actually 'see each other' in the bulk indirectly through the common volume element in the action, while in the models discussed here they only see each other in the boundary. However in the presence of branes with boundaries both fields interact at the branes boundaries [26, 27], so these could mediate a bulk to bulk interaction. It is worthwhile to remark that while 'antipodal symmetry' in Linde's model is postulated ad hoc in the models discussed here it is a natural byproduct of the construction of an action with enhanced symmetry.

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References

- [1] J. Maldacena, JHEP 9807(1998)023, hep-th/9806087.
- [2] J. Maldacena, Adv. Theor. Math. Phys. 2(1998)231, hep-th/9711200, Int. J. Theor. Phys. 38(1999)1113, hep-th/0309246.
- [3] E. Witten, Adv. Theor. Math. Phys. **2**(1998)253, hep-th/9802150.
- [4] G. Horowitz and J. Polchinski, *Gauge/gravity duality*, gr-qc/0602037.
- [5] G.W. Gibbons and S.W. Hawking, Phys. Rev. **D15**(1977)2753.
- [6] S.W. Hawking and D.N. Page, Commun. Mat. Phys. **87**(1983)577.
- [7] M. Henningson and K. Skenderis, JHEP 9807(1998)023, hep-th/9806087.
- [8] R. Emparan, C.V. Johnson and R.C. Myers, Phys. Rev. **D60**(1999)104001, hep-th/9903238.
- [9] V. Balasubramanian and P. Krauss, Commun. Math. Phys. **238**(1999)413, hep-th/9902121.
- [10] S. de Haro, S.N. Solodukhin and K. Skenderis, Commun. Math. Phys. **217**(2001)595, hep-th/0002230.
- [11] K. Skenderis, J. Mod. Phys. A **16**(2001)740, hep-th/0010138.
- [12] I. Papadimitriou and K. Skenderis, JHEP 0508(2005)004, hep-th/0505190.
- [13] P. Mora, R. Olea, R. Troncoso and J. Zanelli, *Vacuum energy in odd-dimensional AdS gravity.*, hep-th/0412046.
- [14] O. Miskovic and R. Olea, Phys. Lett. **B640**(2006)101, hep-th/0603092.
- [15] R. Olea, *Regularization of odd-dimensional AdS gravity: Kounterterms*, hep-th/0610230.
- [16] R. Aros, M. Contreras, R. Olea, R. Troncoso and J. Zanelli, Phys. Rev. Lett. **84**(2000)1647, Phys. Rev. **D62**(2000)044002.
- [17] R. Aros, C. Martínez, R. Troncoso and J. Zanelli, JHEP 05(2002)020, hep-th/0204029.

- [18] J. Zanelli, *Lecture notes on Chern-Simons (super-)gravities*, hep-th/0502193.
- [19] P. Mora, R. Olea, R. Troncoso and J. Zanelli, *Finite Action Principle for Chern-Simons AdS Gravity*, JHEP06(2004)036, hep-th/0405267.
- [20] S. S. Chern, *Complex Manifolds without Potential Theory*, 2nd Ed., (Springer, Berlin, 1979).
- [21] T. Eguchi, P. Gilkey and A. Hanson, Phys. Rept. **66**(1980)213.
- [22] M. Nakahara, *"Geometry, Topology and Physics"*, Adam Hilger, (1991)
- [23] Pablo Mora, *Transgression forms as unifying principle in field theory*, Ph.D.Thesis, Universidad de la República, Uruguay, (2003), hep-th/0512255.
- [24] P. Mora, R. Olea, R. Troncoso and J. Zanelli, *Transgression forms and extensions of Chern-Simons gauge theories*, JHEP0602(2006)067, hep-th/0601081.
- [25] R. Aros, M. Contreras, R. Olea, R. Troncoso and J. Zanelli, *Charges in 2+1 Dimensional Gravity and Supergravity*, presented at the *Strings'99 Conference*, Postdam, Germany, July 1999.
- [26] P. Mora and H. Nishino, Phys. Lett. **B482**(2000)222, hep-th/0002077.
- [27] P. Mora, Nucl. Phys. **594B**(2001)229, hep-th/0008180.
- [28] A. Borowiec, M. Ferraris and M. Francaviglia, J. Phys. **A36**(2003)2589.
- [29] A. Borowiec, L. Fatibene, M. Ferraris and M. Francaviglia, *Covariant Lagrangian Formulation of Chern-Simons and BF Theories*, hep-th/0511060.
- [30] F. Izaurieta, E. Rodriguez and P. Salgado, *On Transgression Forms and Chern-Simons (Super)gravity*, hep-th/0512014.
- [31] R. Aros, *Charges and the boundary in Chern-Simons gravity*, gr-qc/0601120.
- [32] O. Chandía and J. Zanelli, Phys. Rev. **D55**(1997)7580, hep-th/9702025.
- [33] R.B. Mann, Class. Quant. Grav. **14L109**(1997), gr-qc/9607071.
- [34] R. Emparan, Phys. Lett. **B432**(1998)74, hep-th/9804031.
- [35] J. Crisóstomo, R. Troncoso and J. Zanelli, Phys. Rev. **D62**(2000)084013.
- [36] I. Moss, Nucl. Phys. **B729**(2005)179, hep-th/0403106.
- [37] R. Troncoso and J. Zanelli, Int. Jour. Theor. Phys. **38**(1999)1181, hep-th/9807029.
- [38] P. Horava, Phys. Rev. **D59**(1999)046004, hep-th/9712130.

- [39] M. Banados, Phys. Rev. Lett. **88**, 031301 (2002), hep-th/0107214.
- [40] H. Nastase, *Towards a Chern-Simons M theory of $OSp(1|32) \times OSp(1|32)$* , hep-th/0306269.
- [41] A. Linde, Phys. Lett. **B200**(1988)272.
- [42] A. Linde, *Inflation and Quantum Cosmology*, (Academic Press Inc., 1990), see pages 142-146.